International Journal of Theoretical Physics, Vol. 9, No. 4 (1974), pp. 219–224

Correcting the Concept of Denseness for the Bootstrap

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Received: 21 *May* 1973

Abstract

One way of stating the bootstrap theory is to say that hadrons are 'dense'. In order for this to have a truly physical meaning, and in order for the abstract idea of infinite cardinality to be rigorously self-consistent, it is necessary to redefine the concept of denseness to mean that size is increasing as velocity.

1. *Introduction*

Using the axioms of formal logic, the present author has recently shown (Allen, 1973) that the bootstrap theory of hadronic constituents is logically superior to the traditional theory favored by fundamentalists. This conclusion arises from the observation that the fundamentalness of a particle can never be demonstrated experimentally: given a purportedly fundamental particle which has not yet been split, one can always argue that the particle could be split if more energy were available for that purpose. On the other hand, the latter argument need not be demonstrated experimentally under the axiom of formal logic that if condition p is false, then the implication $p \rightarrow q$ must be true

Evidently, some physicists will insist on talking about the constituents of hadrons under any circumstances (Lubkin, 1972). This is not inconsistent with the present author's results, provided that one always talks about such constituents in the *plural.* In other words, the bootstrap theory may be cast as the following proposition.

Proposition 1.1. *Hadrons are 'dense' in the mathematical sense of the word.*

Indeed, the mathematical concept of a dense set and the physical concept of the bootstrap are so intimately related that the latter theory cannot be formalized with any rigor until the former, purely mathematical issue has been addressed. One is thus faced with the problem of attaching some realistic physical meaning to the idea of uncountably infinite sets. At least for the pragmatic physicist, it should seem less than satisfactory to interpret Proposition 1.1 to mean that hadrons are composed of an infinitely large number of infinitely small parts.

Interestingly, the idea of infinite cardinality first arose in dealing with the physical problems of space, time, and motion as set forth in the Zeno paradoxes (Ballif & Dibble, 1969). Because much has been learned about space, time, and motion since the days of Zeno and since Cantor formalized the concept of an infinite set (Cantor, 1915), it is not surprising that mathematical formalisms which continue to rely on these ancient intuitions can be shown to lack rigor (Allen, 1972). Hence the approach here will be to return to the original intuitions underlying the classical concept of the continuum, and revise them so that the idea of denseness will be both physically meaningful and internally self-consistent. If this seems to be an overly pretentious aspiration, note that the current use of nonfinite mathematics invites the same sort of criticism that Steigman has offered with respect to antimatter: 'Scientists should not postulate the existence of antimatter to solve a problem in astrophysics, and then use the solved problem to prove that antimatter exists' (Thomsen, 1973).

2. *The New Denseness*

The classical concept of denseness may be stated in the following form.

Axiom 2.1. *A point-sized object at rest occupies one location of space, dx in size. If the object is set in motion, then it sequentially occupies an infinite succession of such locations, each being dx in size.*

The perspective of Axiom 2.1, which is based on the paradigms of Zeno, Newton, and Cantor, is clearly anticlinal to the thrust of modern physics. For one thing it depends upon absolute space and absolute certainty of position, contrary to the first principles of relativistic and quantum mechanics, respectively. The updated axiom proposed here is more in accord with modern theory and may be stated as follows.

Axiom 2.2. *Spacial size is a monotone increasing function of velocity, scaled by a nonvanishing interval of time. Thus a point-sized object has a proper size of dx which is transformed by motion to $d^*x > dx$, where the specific solution for d^*x is stochastic and depends upon the parameters of the particular problem at hand.*

Note that the new axiom is antithetical to, but separate from, Lorentz contraction. The rationale underlying Axiom 2.2 is that the spacial size of an object is measured by the volume of space in which the probability of interacting with it kinematically is unity. Suppose then that a point of mass moves through a set of spacial co-ordinates $\{x_i, y_i, z_i\}$ over the bounded interval of time $a \leqslant t \leqslant b$. Taking the foregoing definition of spacial size in the most general sense, it follows that over the interval of time $a \leqslant t \leqslant b$, the point of mass has a size that is equal to the total volume given by the set of co-ordinates $\{x_i, y_i, z_i\}$. This interpretation of size differs from Axiom 2.1 in two important respects which will be treated separately.

First, the classical view is that the interval of time over which the spacial size of a moving object is measured must be 'instantaneous', and this is defined such that the resulting measurement must be equal to the size of the object when measured at rest. In other words, an instantaneous measurement of size is defined so as to negate the motion of objects. This is not supported by phenomena since in order to perform any measurement on an object some apparatus must interact with it, and at the very *instant* this interaction takes place a moving object will be found to be quite different from an object at rest, i.e., it will have some kinetic energy as a result of its motion. Under Axiom 2.2, on the other hand, the question of instantaneous measurement must be considered from a different perspective. Consider two points of mass, α and β, having uniform velocities v_α and v_β such that

$$v_\alpha > v_\beta \tag{2.1}$$

Over a fixed interval of time Δt, the spacial size of these two objects is now deemed to be

$$\begin{aligned} s_\alpha &\propto v_\alpha \Delta t \\ s_\beta &\propto v_\beta \Delta t \end{aligned} \tag{2.2}$$

For the purpose of having an 'instantaneous' measurement, one may let Δt become arbitrarily small to assume the value dt. However, since the momentum of an object is not to vanish while it is moving, and since to preserve analyticity one must be able to divide by dt, dt must remain positive even when indefinitely small. As a result, the monotone increasing relationship between size and velocity given in (2.2) is maintained for 'instantaneous' measurements. Hence, it follows from inequality 2.1 that the size of α is greater than the size of β, even for instantaneous measurements.

Secondly, the classical viewpoint of size is intimately related to the idea that moving objects can maintain a fixed, definite spacial location. While this idea may be viable for macroscopic objects, it is contrary to quantum mechanical principles when applied to point-sized objects. As just shown, Axiom 2.2 defines the position of a point-like object to be much more stochastic. In particular, the new axiom carries the idea that a particle is where it goes. Thus one is forced to talk about the location of a particle in fuzzy rather than binary terms, thereby admitting the axiomatic necessity for de Broglie waves.

3. *The New Rigor*

An important property of the classical infinite set G is that its elements may be put into one-to-one correspondence with the elements of its *proper* subset $\mathscr{G}$. This isomorphism is demonstrated through some scheme which pairs any unique but arbitrary element of G with a unique, corresponding element of $\mathscr{G}$. In order to demonstrate the contradictions which typically accompany such schemes and the way in which the use of Axiom 2.2 in lieu of Axiom 2.1 can produce new and self-consistent results, it is convenient to consider a geometric example widely used in undergraduate textbooks (see for example, Boyer, 1955; Kasner & Newman, 1940; Richardson, 1955; and Spiegel, 1969).

This example employs a triangle having base B and apex L. Within the triangle is a line segment B' which is parallel to B, but closer to L than is B.

Clearly, B' must be shorter than B and thus, in length, is a proper subset of B. In addition, the triangle contains a line segment R_i which intersects L and an arbitrary point on B, thus intersecting B' as well. R_i may be used to pair a point on B with a point on B' under the following Euclidean postulate.

Postulate 3.1. *$\{B \cap R_i\}$ and $\{B' \cap R_i\}$ are both one-element sets.*

Purportedly, the conclusion that B maps onto B' follows immediately from Postulate 3.1 on the grounds that R_i is defined such that it can be anywhere within the area of the triangle. This thinking, however, is not very rigorous. Since B and B' each contain more than just one point, B cannot be mapped onto B' with just one line R_i because this only maps one element of B onto one element of B'. Rather, a family of such lines $\{R_i\}$ must be used. Thus the conclusion that B maps onto B' depends upon and implies the following additional postulate which is not one of the Euclidean postulates.

Postulate 3.2. *As the number of lines in $\{R_i\}$ grows without bound, B is mapped onto B'.*

But Postulate 3.2 is false, as can be easily demonstrated. Even if $\{R_i\}$ is dense its elements are still ordered in space. That is, given some $R_i \in \{R_i\}$, there exists a unique $R_{i+1} \in \{R_i\}$ which is closer to R_i than any other element of $\{R_i\}$ in a given direction. Thus if $\{R_i\}$ maps onto the area of the triangle, then R_i and R_{i+1} alone must map onto the uniformly continuous area of the triangle bounded by R_i and R_{i+1}. But since R_i and R_{i+1} intersect at L, the distance between them is positive except at L. Indeed, it is monotone increasing as distance from L. Hence there is an ever-widening gap between R_i and R_{i+1} which leaves that part of the triangle bounded by these two lines uncovered.

There is another approach to the problem of mapping B onto B' which addresses the original dynamical intuitions embodied in the Zeno paradoxes. This approach consists of replacing $\{R_i\}$ with a radius R having its locus at L, which spins across the area of the triangle. The classical assumptions underlying this dynamical model (as exemplified by the calculus) are the same as before since, classically, dynamical systems are modeled with essentially the same intuitions as nondynamical systems. Specifically, the present situation is classically modeled by the following analogies for Postulates 3.1 and 3.2, the first of which stems from Axiom 2.1.

Postulate 3.3. *At any instant during which R is within the area of the triangle, it intersects B at exactly one point and likewise intersects B' at exactly one point; the instantaneous position of the spinning radius is identical to a position which it could assume when not moving.*

Postulate 3.4. *If R moves completely across B in t time, then R moves completely across B' in t time.*

Postulate 3.4 is beyond questioning since it can be easily verified experimentally. For example, if this were not true, then rods could not spin without breaking up into pieces and clocks could not have hands. Indeed, the obvious truth of Postulate 3.4 is probably responsible for the intuitive acceptance of Postulate 3.2 under the nondynamical circumstances in which the postulate leads to a contradiction. Postulate 3.3, on the other hand, is now a Euclidean postulate which has been transported into the realm of dynamical systems contrary to its intended purpose. As a result, the previous contradiction continues to appear. By Postulate 3.3 the set of all positions R assumes in spinning across the triangle must be a dense set of lines, but one in which any two consecutive lines are separated by a positive distance (increasing as distance from L) because, again, they all meet at L thus leaving part of the triangle uncovered as before, contrary to Postulate 3.4. In order to eliminate this self-contradiction, let the Euclidean Postulate 3.3 be replaced in this dynamical situation by an alternative, dynamical postulate which may be easily verified experimentally, and which substitutes Axiom 2.2 for Axiom 2.1.

Postulate 3.5. *Over any positive interval of time Δt, the distance a point on R moves is proportional to its distance from L, and this proportional relationship holds as Δt becomes arbitrarily small to assume the value dt.*

Under Postulate 3.5 the previous contradiction no longer appears because the 'instantaneous' position of R is now longer at B than at B'. In other words, once spinning, the radius is transformed into a circle's sector. As noted previously, another way of interpreting this transformation is to say that the spinning radius has wave properties. Consider a point on the radius r distance from L. A frequency is inherently associated with such a point, $\nu = t^{-1}$, where t is the time required for R to make one complete revolution. The speed of the point is then required to be $v(r) = 2\pi r\nu$. Hence the point's 'wavelength' must be defined as $\lambda = 2\pi r$ in order to preserve the relationship $v(r) = \lambda\nu$. On what grounds might this definition be justified? Is the instantaneous position of the point *really* distributed stochastically over the circumference of its orbit? In order to answer these questions, let the point have a mass of m and an angular momentum of $\hbar$. Then, by defining λ as $2\pi r$, one can obtain it through de Broglie's equation. The proof is trivial:

$$\text{Assume } \hbar = rmv(r) \tag{3.1}$$

Divide both sides of (3.1) by $mv(r)$ to obtain

$$\hbar/mv(r) = r \tag{3.2}$$

Next multiply both sides of (3.2) by 2π to obtain

$$h/mv(r) = 2\pi r \tag{3.3}$$

Under the definition $\lambda = 2\pi r$, (3.3) becomes de Broglie's equation

$$h/mv(r) = \lambda$$

What more could one ask?

4. *Conclusion*

Under the new Axiom 2.2, one can cast the bootstrap theory very conveniently as Proposition 1.1. To say that hadrons are dense no longer means that they have an infinitely large number of infinitely small constituents, as would be true under the classical Axiom 2.1. Rather, it now means that what a hadron is composed of varies according to the motion of the hadron at the time this question is investigated. In other words, as the bootstrap theory says, there are as many hadrons as there are self-consistent states for hadrons to assume.

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